Lecture 2: Girth, Connectivity and Bipartite Graphs

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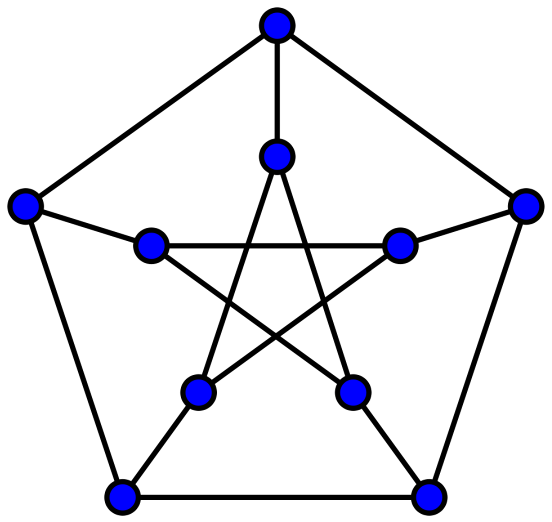
https://shuaili8.github.io/Teaching/CS3330/index.html

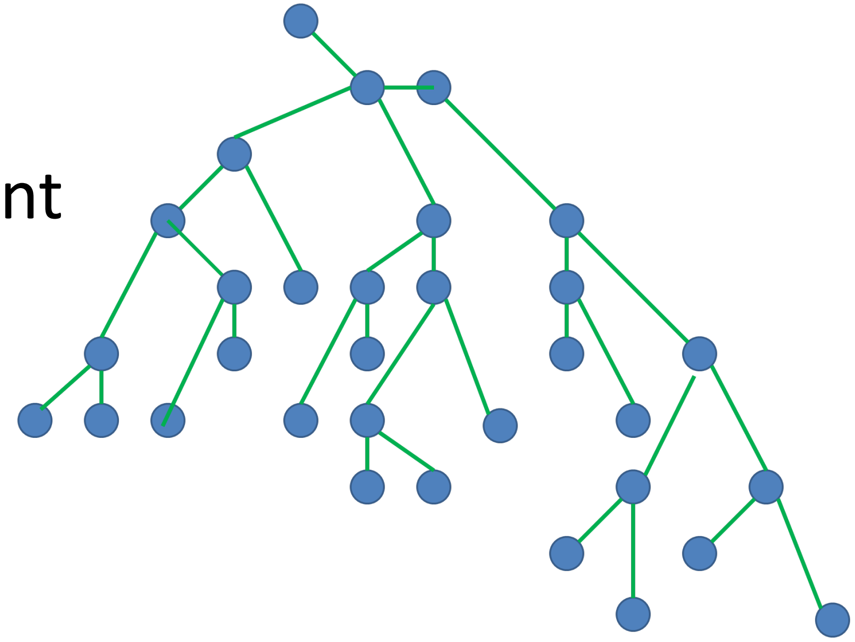
Girth

- The minimum length of a cycle in a graph G is the girth $g(G)$ of G
- Example: The Peterson graph is the unique 5-cage
  - cubic graph (every vertex has degree 3)
  - girth = 5
  - smallest graph satisfies the above properties



Girth (cont.)

- A tree has girth ∞
- Note that a tree can be colored with two different colors
- $\Longrightarrow$ A graph with large girth has small chromatic number?
- Unfortunately NO!
- Theorem (Erdős, 1959) For all k, l, there exists a graph G with $g(G) > l$ and $\chi(G) > k$



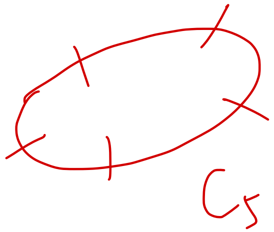
Girth and diameter

- Proposition (1.3.2, D) Every graph G containing a cycle satisfies $g(G) \le 2\,\mathrm{diam}(G) + 1$

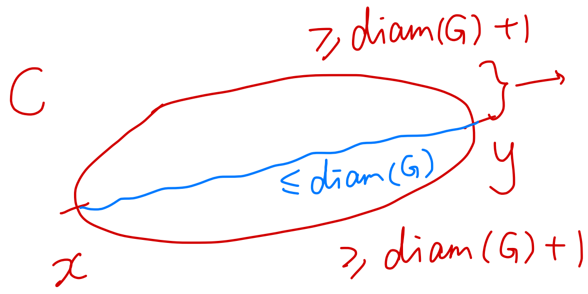
反证法

Suppose $g(G) \geq 2\,\mathrm{diam}(G) + 2$

- When the equality holds?



C_{2n+1}



$\rightarrow$ closed walk of length $\leq g(G) - 1$

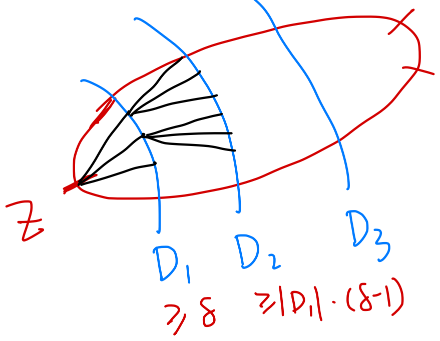
$\Rightarrow \exists$ cycle of length $< g(G)$

Contradiction!

Girth and minimal degree lower bounds graph size

- $n_0(\delta, g) := \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i, & \text{if } g = 2r + 1 \text{ is odd} \\ 2 \sum_{i=0}^{r-1} (\delta - 1)^i, & \text{if } g = 2r \text{ is even} \end{cases}$
- Exercise (Ex7, ch1, D) Let G be a graph. If $\delta(G) \geq \delta \geq 2$ and $g(G) \geq g$, then $|G| \geq n_0(\delta, g)$
- Corollary (1.3.5, D) If $\delta(G) \geq 3$, then $g(G) < 2\log_2|G|$ $\Leftrightarrow 2^{g/2} < |G|$

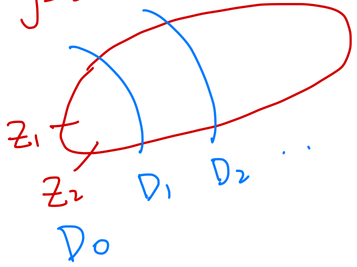
$g = 2r+1$ odd



Claim: $\forall x \in D_1$, $N(x) - \{z\} \subseteq D_2$

$y \in D_1$ $(N(x) - \{z\}) \cap (N(y) - \{z\}) = \phi$

$g = 2r$ even



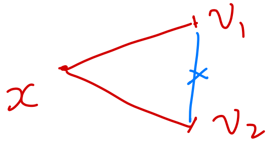
r $\quad \Delta \quad$ $|G| \leq \cdots$

g $\quad \delta \quad$ $|G| \geq \cdots$

Triangle-free upper bounds # of edges

- Theorem (1.3.23, W, Mantel 1907) The maximum number of edges in an n-vertex triangle-free simple graph is $\lfloor n^2/4 \rfloor$

Take a vertex w/ maximum degree Δ



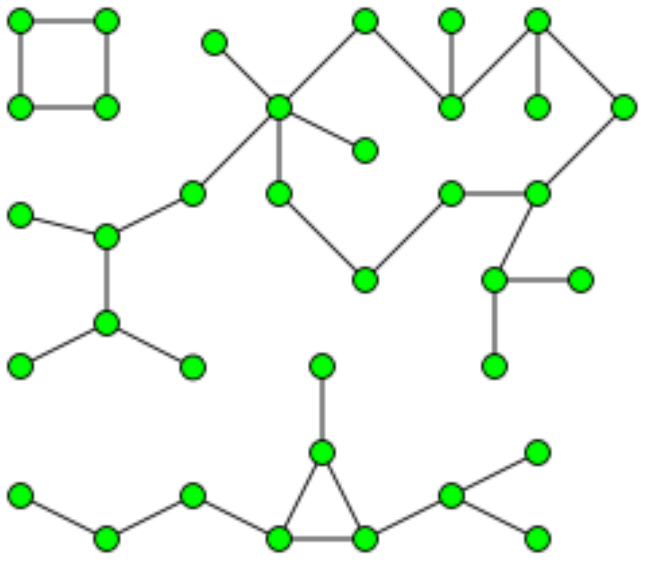
Every edge has an endpoint $\notin N(x)$

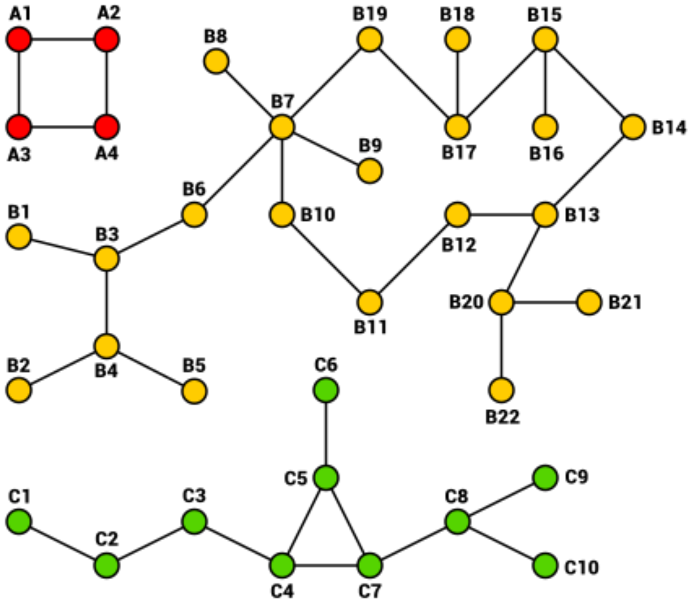
$\Rightarrow |E| \leq \sum_{v \notin N(x)} d(v) \leq \Delta(n-\Delta) = -\Delta^2 + n\Delta \leq \frac{n^2}{4}$

- The bound is best possible
- There is a triangle-free graph with $\lfloor n^2/4 \rfloor$ edges: $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$
- Extremal problems

Connected, connected component

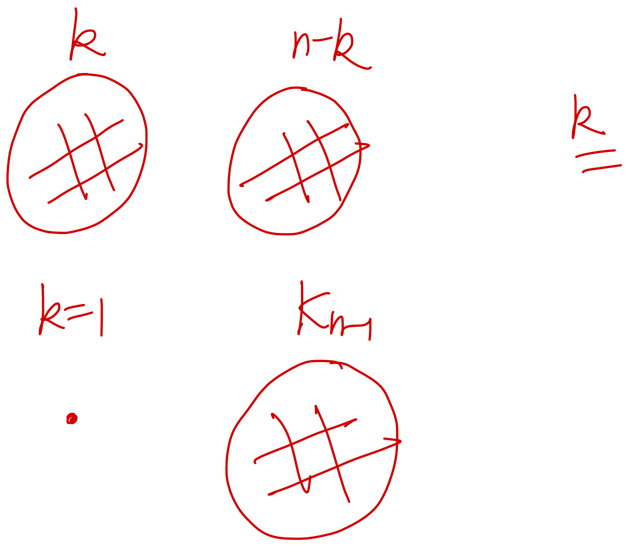
- A graph G is connected if $G \neq \emptyset$ and any two of its vertices are linked by a path
- A maximal connected subgraph of G is a (connected) component





Quiz

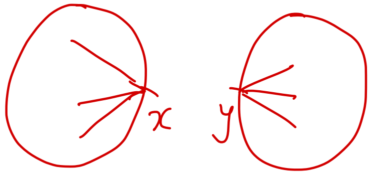
- Problem (1B, L) Suppose G is a graph on 10 vertices that is not connected. Prove that G has at most 36 edges. Can equality occur?
- More general (Ex9, S1.1.2, H) Let G be a graph of order n that is not connected. What is the maximum size of G?



Connected vs. minimal degree

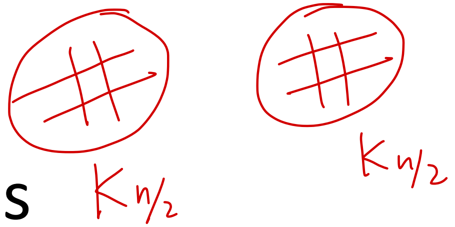
- Proposition (1.3.15, W) If $\delta(G) \geq \frac{n-1}{2}$, then G is connected

反证法



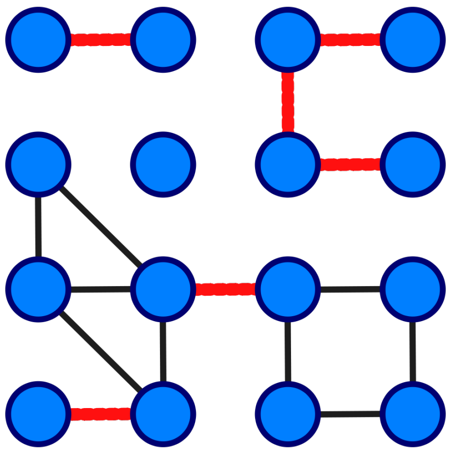
$|G| \geq d(x)+1 + d(y)+1$
$\geq 2\delta(G)+2 > n$ Contradiction!

- (Ex16, S1.1.2, H; 1.3.16, W)
  If $\delta(G) \geq \frac{n-2}{2}$, then G need not be connected

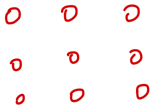


- Extremal problems
- “best possible” “sharp”

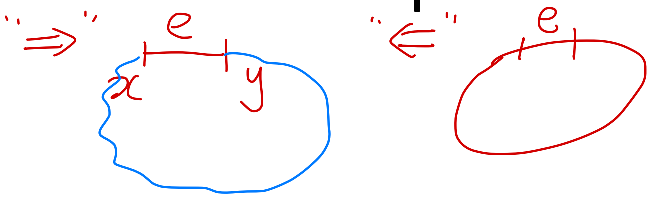
Add/delete an edge



- Components are pairwise disjoint; no two share a vertex
- Adding an edge decreases the number of components by 0 or 1
  - $\Rightarrow$ deleting an edge increases the number of components by 0 or 1
- Proposition (1.2.11, W)

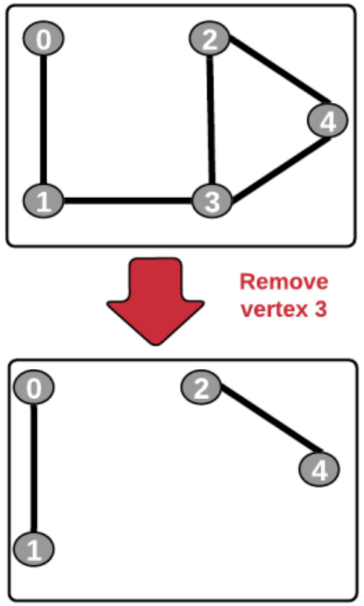


  Every graph with n vertices and k edges has at least $n-k$ components
- An edge e is called a bridge if the graph $G-e$ has more components
- Proposition (1.2.14, W)



  An edge e is a bridge $\Leftrightarrow$ e lies on no cycle of G
  - Or equivalently, an edge e is not a bridge $\Leftrightarrow$ e lies on a cycle of G

Cut vertex and connectivity

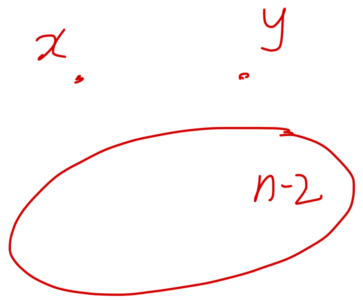


- A node v is a cut vertex if the graph $G - v$ has more components
- A proper subset S of vertices is a vertex cut set if the graph $G - S$ is disconnected, or trivial (a graph of order 0 or 1)
- The connectivity, $\kappa(G)$, is the minimum size of a cut set of G
  - The graph is k-connected for any $k \leq \kappa(G)$

Connectivity properties

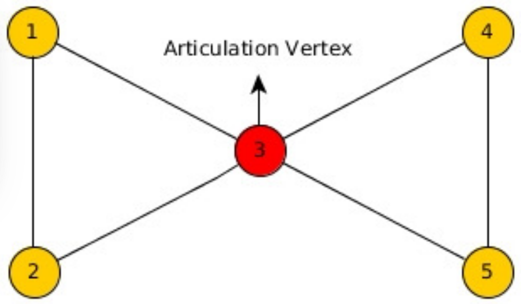
- $\kappa(K^n) = n - 1$
- If G is disconnected, $\kappa(G) = 0$
  - $\Rightarrow$ A graph is connected $\Leftrightarrow \kappa(G) \geq 1$
- If G is connected, non-complete graph of order n, then

$$1 \leq \kappa(G) \leq n - 2$$



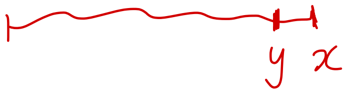
Connectivity properties (cont.)

Proposition (1.2.14, W)
An edge e is a bridge $\Leftrightarrow$ e lies on no cycle of G
- Or equivalently, an edge e is not a bridge $\Leftrightarrow$ e lies on a cycle of G



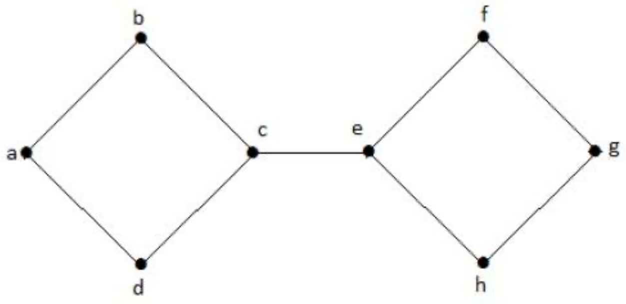
- $\kappa(G) \geq 2 \Leftrightarrow G$ is connected and has no cut vertices
- A vertex lies on a cycle $\not\Rightarrow$ it is not a cut vertex
  - $\Rightarrow$ (Ex13, S1.1.2, H) Every vertex of a connected graph G lies on at least one cycle $\not\Rightarrow \kappa(G) \geq 2$
  - (Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies G has at least one cycle

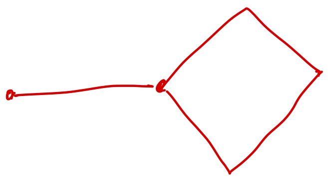
反证法.
G has no cycle
Take the longest path
$d(x)=1$



$\Rightarrow y$ is a cut vertex
Contradiction!

- (Ex12, S1.1.2, H) G has a cut vertex vs. G has a bridge



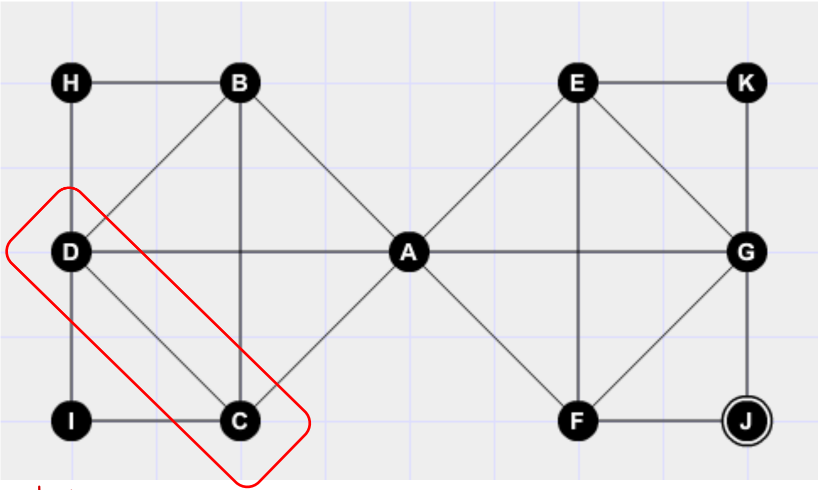


Connectivity and minimal degree

- (Ex15, S1.1.2, H)

(a) $\kappa(G) \leq \delta(G)$

(b) If $\delta(G) \geq n-2$, then $\kappa(G) = \delta(G)$



(a) · G complete graph. $\delta(G)=n-1$, $\kappa(G)=n-1$

· G non-complete

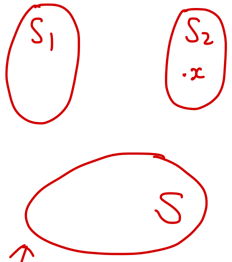
$\exists v \quad d(v)=\delta(G)<n-1$

$N(v)$ separates v and $G-\{v\}-N(v)$

(b) · $\delta(G)=n-1$. G is complete ✓

· $\delta(G)=n-2 \quad \kappa(G)\leq n-2 \quad$ "="

Assume $\kappa(G)\leq n-3$



vertex cut set of size $\kappa(G)$

$|S_1 \cup S_2| \geq 3$

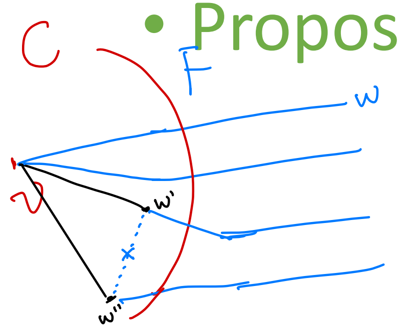
w.l.o.g. $|S_1|\geq 2$

$d(x) \leq |S_2|-1+|S| = n-|S_1|-1 \leq n-3$

Contradiction!

Edge-connectivity

- A proper subset $F \subset E$ is edge cut set if the graph $G - F$ is disconnected
- The edge-connectivity $\lambda(G)$ is the minimal size of edge cut set
- $\lambda(G) = 0$ if G is disconnected
- Proposition (1.4.2, D) If G is non-trivial, then $\kappa(G) \le \lambda(G) \le \delta(G)$



$\cdot\, vw \in F$

$\cdot\, vw', vw'' \notin F$ $\in C$

1° $\exists$ v not adjacent to F

C is the component of v in $G-F$

$S :=$ vertices in C incident to F

$|S| \le |F|$

2° Every v is incident to F

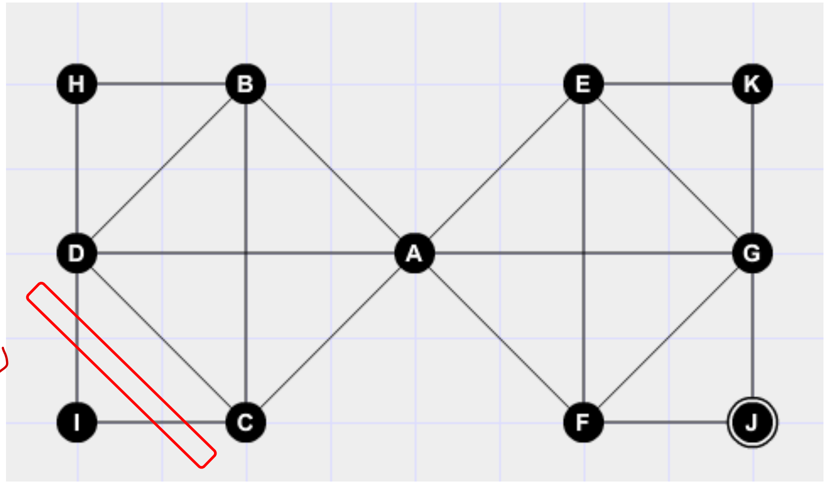
$d(v) = |N(v)| \le |F| = \lambda(G)$

G is complete graph ✓

G is not complete

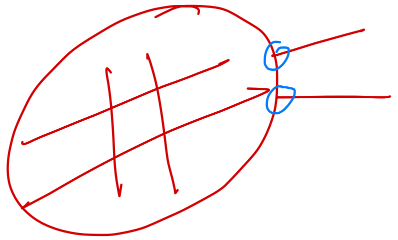
Take v w/ $d(v) < n-1$

$N(v)$ separate v and $G - \{v\} - N(v)$

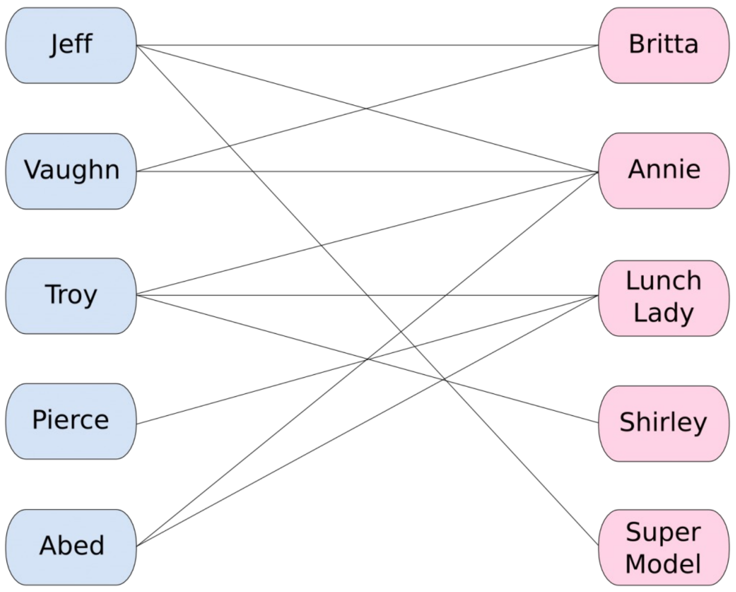


Large average (minimal) degree implies local large connectivity

- Theorem (1.4.3, D, Mader 1972) Every graph G with $d(G) \geq 4k$ has a $(k+1)$-connected subgraph H such that $d(H) > d(G) - 2k$.



Bipartite graphs



- **Theorem** (1.2.18, W, Kőnig 1936)
  A graph is bipartite ⟺ it contains no odd cycle

"⇒" ✓

Proposition (1.2.15, W) Every closed odd walk contains an odd cycle

"⇐" w.l.o.g. G is connected

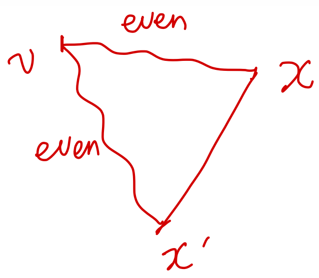
Take $v \in G$ $\quad X = \{x \in G : d(x,v) \text{ is even}\}$

$Y = \{ \dots \quad \text{odd}\}$

$\forall x, x' \in X$ (or Y)

Suppose $xx' \in E(G)$

$\begin{cases} v = x, \; vx' \in E \quad d(v,x') = 1 \text{ not even } \times \\ v \neq x, x', \end{cases}$

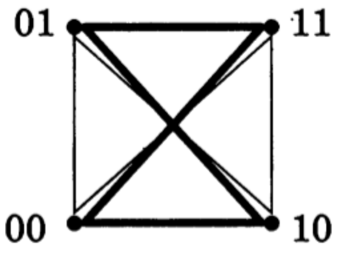


← closed odd walk

Complete graph is a union of bipartite graphs

- The union of graphs $G_1, \ldots, G_k$, written $G_1 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$

- Consider an air traffic system with k airlines
  - Each pair of cities has direct service from at least one airline
  - No airline can schedule a cycle through an odd number of cities
  - Then, what is the maximum number of cities in the system?



- Theorem (1.2.23, W) The complete graph K_n can be expressed as the union of k bipartite graphs $\Leftrightarrow n \leq 2^k$

Induction on k

o $k=1$ $\quad n \leq 2$ ✓

$n \geq 3$ odd cycle ×

o $k-1$ $(k \geq 2)$ ✓

To prove k

B_k is a bipartite graph on X, Y, $|X \cup Y| = n$

"$\Rightarrow$" $K_n = B_1 \cup \cdots \cup B_k$

$K_X \leq B_1 \cup \cdots \cup B_{k-1}$

$K_Y \leq \cdots$

$\therefore |X|, |Y| \leq 2^{k-1}$ $\quad \therefore n \leq 2^k$

"$\Leftarrow$" Divide $[n]$ into X and Y. $|X|, |Y| \leq 2^{k-1}$

$K_X = B_1 \cup \cdots \cup B_{k-1}$

$K_Y = B_1' \cup \cdots \cup B_{k-1}'$

$K_n = (B_1 \cup B_1') \cup \cdots \cup (B_{k-1} \cup B_{k-1}') \cup K_{X,Y}$

Bipartite subgraph is large

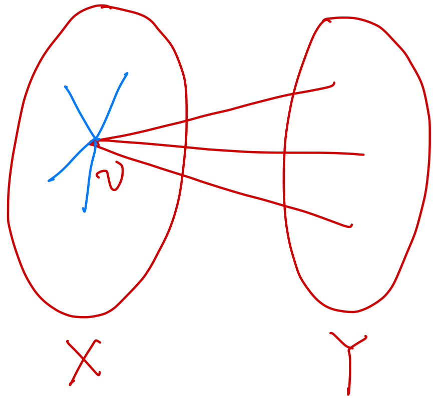
- **Theorem** (1.3.19, W) Every loopless graph G has a bipartite subgraph with at least $|E|/2$ edges

Start w/ any partition X, Y of G

$H = (X \cup Y, E(X, Y))$

if $\forall v \in V(H) = V(G)$, $d_H(v) \geq \frac{1}{2} d_G(v) \Rightarrow |E(H)| \geq \frac{1}{2} |E(G)|$

if $\exists$ v s.t. $d_H(v) < \frac{1}{2} d_G(v)$



Move v to the other side

It will strictly increase $|E(H)|$

Thus the process will terminate

Summary

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https://shuaili8.github.io

- Girth
  - Girth vs diameter
  - Girth and minimal degree lower bounds graph size
  - Girth > 3 upper bounds # of edges
- Connectivity
  - Connected components
  - Bridge/cut vertex/connectivity/edge-connectivity
  - Minimal degree and connectivity
  - $\kappa(G) \leq \lambda(G) \leq \delta(G)$
  - Large average (minimal) degree implies local large connectivity
- Bipartite graphs
  - Equivalent to containing no odd cycle
  - Every graph can be decomposed as a union of bipartite graphs, with one large enough

Questions?